

On the difference between the charge-free and the charge-neutral solutions of Maxwell equations

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Abstract

It is conventionally believed that solutions of so called “free” Maxwell equations for $\rho = 0$ (density of charge) describe the free electromagnetic field in empty space (if one considers the free field as a field, whose flux lines neither *begin* nor *end* in a charge). We consider three types of regions: (i) “isolated charge-free” region (where all electric fields, generated by charges outside that particular region, *are zero*), for example, inside a hollow conductor of any shape or in a free-charge Universe; (ii) “non-isolated charge-free” region (where all electric fields, generated by charges outside that particular region, *are not zero*) and (iii) “charge-neutral” region (where point charges exist but their algebraic sum is zero). The paper notes that there are two families of solutions: (1) In “isolated charge-free” regions electric *free field* does not exist *in the context* of Maxwell’s equations, but there may exist a *time-independent* background magnetic field. (2) In both “charge-neutral” and “non-isolated charge-free” regions where the homogeneous condition $\rho = 0$ also holds, Maxwell’s equation for electric field have non-zero solution, as in the conventional view, but this solution is not *free field*. We mention some implications related to free-electromagnetic fields and the simplest charge-neutral universe.

I. INTRODUCTION

It is well-known that the set of four Maxwell’s equations (ME) [1,2] describes different phenomena according to particular initial and boundary conditions (BC). The authors of this note have independently found that the set of solutions of ME may be larger than conventionally believed [3-5]. As part of the process to establish BC for our generic problem, we explore here the meaning of the solutions of ME in regions of space with null charge density ($\rho = 0$).

Conventionally, $\rho = 0$ represents “empty space” (see, e.g., Purcell [1], page 331). Under this condition, both Eqs. (3) and (4) (see below) describe solenoidal fields, which imply that the electric and magnetic fields (**E** and **H**) in that region of space are transverse to the *instantaneous* [6] direction of propagation. Moreover, since there are no charges in such region, the electromagnetic wave corresponds to a free field, whose flux lines neither begin nor end in a charge.

We want to argue here that such long-standing interpretation is not completely consistent with the physics behind ME. The remainder of this note is organized as follows:

in Section II we critically revisit the conventional interpretation to find that $\varrho = 0$ leads to two families of solutions. Section III explores some implications of our findings and Section IV closes the paper.

II. THE CONVENTIONAL INTERPRETATION CRITICALLY REVISITED

In CGS units, Maxwell's equations are¹

$$\nabla \cdot \mathbf{E} = 4\pi\varrho, \quad (1)$$

$$\nabla \cdot \mathbf{H} = 0, \quad (2)$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}. \quad (4)$$

Charge conservation is assured by the standard continuity condition:

$$\nabla \cdot \mathbf{j} + \frac{\partial \varrho}{\partial t} = 0. \quad (5)$$

We consider three types of regions: (i) “isolated charge-free” region (where all electric fields, generated by charges outside that particular region, *are zero*), for example, inside a hollow conductor of any shape or in a free-charge Universe; (ii) “non-isolated charge-free” region (where all electric fields, generated by charges outside that particular region, *are not zero*) and (iii) “charge-neutral” region (where point charges exist but their algebraic sum is zero). Usually, one set² $\varrho = 0$ in (1) and (3) at the *whole* space (or in “isolated charge-free” region, see (i)) and obtains *free* equations for *free* field. We argue here that this operation does not lead to free-field solution of ME. For our argument, it is important to recall a process of obtaining Eqs. (1) and (3).

We know that Gauss' law claims [1]: *The flux of the electric field \mathbf{E} through any closed surface, that is, the integral $\oint \mathbf{E} \cdot d\mathbf{a}$ over the surface, equals 4π times the total charge enclosed by the surface:*

¹Recently it was shown [4] that one has to use total time-derivatives in (1)-(5) but one can do not attach importance to this here. Recall that $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$ in vacuum in CGS units.

²in (3) $\varrho\mathbf{V} = \mathbf{j}_{cond}$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = 4\pi Q = 4\pi \sum_i q_i = 4\pi \int_V \rho dv. \quad (6)$$

We call this statement a *law* because it is *equivalent* to Coulomb's law and it could serve equally well as the basic law of electrostatic interactions, *after* charge and field have been *defined*. Gauss' and Coulomb's laws are not two independent physical laws, but the same law expressed in different ways. Looking back over a proof of Eq.(6) in any textbook, we can see that it *hinged* on the *inverse-square nature* of the interaction. Thus this theorem (law) is applicable *solely* to inverse-square field in physics. We stress several aspects:

a) Coulomb's law is defined in terms of the individual q_i , so that the expression for charge Q (Eq.(6)) in terms of charge density ρ is only strictly valid as a limit when a very large number of charges is present. (Before we are criticized, we hasten to add that, of course, ρ may be treated as δ -function).

b) Gauss' law *only* applies to inverse-square fields, but they do not need to be isotropic. Hence, it contains Coulomb's law, but it is somewhat more general ([1]. p.24).

c) The right-hand-sides of Eq.(6) may be zero in two different ways: (*) **Charge-free condition**, $Q = 0$ when $q_i = 0$, all " i ". (**) **Charge-neutral condition**, $Q = 0$ when $q_i \neq 0$, all " i " independently.

Evidently, there is no reason to expect that \mathbf{E} on the *lhs* of Gauss' law (6) should be the same for cases (*) and (**) above. Indeed, for an *isolated* charge-free region the *only* solution is

$$\mathbf{E} = 0, \quad (7)$$

which simply means that a non-existing charge cannot produce an electric field. Note that previous assertion is *qualitatively* different to saying that there exist an electric field in the region that becomes zero when $Q = 0$.

Let us remember now the Ostrogradsky-Gauss' theorem. If this theorem holds for any vector field, it certainly holds for \mathbf{E} :

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} dv. \quad (8)$$

Both Eq.(6) and Eq.(8) hold for any volume we care to choose - of any shape, size, or location. Comparing them, we see that this can only be true if at *every* point,

$$\nabla \cdot \mathbf{E} = 4\pi\rho. \quad (9)$$

But we *always* must take into account that (because of origin of (6)!) if ρ is zero in the *isolated* charge-free region, $\nabla \cdot \mathbf{E}$ must be zero because of \mathbf{E} is zero in the same *isolated* charge-free region.

Now we can recall the origin of Eq.(3). Really, Maxwell found his famous paradox (because of equation of continuity (5)):

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j}_{cond} + (?) \quad (10)$$

and discovered what $(?)$ must be:

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j}_{tot} = \frac{4\pi}{c} (\mathbf{j}_{cond} + \mathbf{j}_{disp}), \quad (11)$$

$$\nabla \cdot \mathbf{j}_{tot} = \nabla \cdot \mathbf{j}_{cond} + \nabla \cdot \mathbf{j}_{disp} = 0, \quad (12)$$

$$\nabla \cdot \mathbf{j}_{disp} = -\nabla \cdot \mathbf{j}_{cond} = \frac{\partial \rho}{\partial t}. \quad (13)$$

Using (9) one obtains:

$$\nabla \cdot \mathbf{j}_{disp} = \frac{1}{4\pi} \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = \nabla \cdot \left(\frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \right). \quad (14)$$

General solution of this equation is

$$\mathbf{j}_{disp} = \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \{\mathbf{F}_1(x, y, z, t)\} + \mathbf{F}_2(t) + \mathbf{const}. \quad (15)$$

Maxwell (and others, following him) set the terms

$$\nabla \times \{\mathbf{F}_1(x, y, z, t)\} + \mathbf{F}_2(t) + \mathbf{const} = 0 \quad (16)$$

and as a results obtains Eq.(3). But *after* (**attention!**) obtaining Eq.(3) Maxwell *at al* set following:

In empty space, the terms with ρ and $\mathbf{j}_{cond} = \rho \mathbf{V}$ are zero, and Maxwell's equations become

$$\nabla \cdot \mathbf{E} = 0 \quad (17)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (18)$$

$$\nabla \times \mathbf{H} = +\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (19)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}. \quad (20)$$

But if we set $\varrho = 0$ at *whole* space we must consider now (see point (c) above) the **charge-free condition** when $q_i = 0$ for all i . In other words, we obtain *isolated* charge-free region. And as a result we must obtain for this region

$$\nabla \cdot \mathbf{E} = 0 \quad (21)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (22)$$

$$\nabla \times \mathbf{H} = (?) \quad (23)$$

$$\nabla \times \mathbf{E} = 0 \quad (24)$$

because \mathbf{E} has to be zero in every point of this region³. Of course, in the *non-isolated* charge-free region and in the charge-neutral region Eqs. (17)-(20) keep their form but in this case we also cannot obtain an electric *free* field, whose flux lines *neither begin nor end in a charge* (recall the origin of Eq.(9)!).

We turn now to some implications of our interpretation.

III. IMPLICATIONS OF OUR INTERPRETATION

A. Isolated charge-free region

Consider an isolated region \mathcal{R}_0 where no charges are present, i.e. $\mathcal{Q} = 0$, $\varrho = 0$ everywhere. Eq. (7) applies, so that $\mathbf{E} = 0$ everywhere in the whole space spanning \mathcal{R}_0 . Assuming that Maxwell's equations are valid in \mathcal{R}_0 it follows that magnetic field \mathbf{H} may still exist, because Maxwell's Eq. (2) is completely independent of ϱ . Indeed, in addition to the trivial solution $\mathbf{H} = 0$, many other solutions of $\nabla \cdot \mathbf{H} = 0$ are possible. For instance, $\mathbf{H} = H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k}$ with $H_x = F_x(ct; y, z)$, $H_y = F_y(ct; x, z)$, $H_z = F_z(ct; x, y)$.

³the sense of (?) in Eq.(23) we explain in Subsection A of Section III

In a charge-free region Faraday's equation (4) reduces to $\frac{\partial \mathbf{H}}{\partial t} = 0$, hence \mathbf{H} is time-independent. Our generic solution thus becomes $B_x = F(y, z)$, $B_y = F(x, z)$ and $B_z = F(x, y)$, where we have noted that in isotropic region there is no reason for the functional dependence to be different along arbitrary orientations.

Finally, Ampère's law (3) leads (see Eq.(15)) to

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j}_{\text{mag}} \quad (25)$$

where \mathbf{j}_{mag} may be some *magnetic* displacement current density. Eq. (25) does not impose further constraints onto \mathbf{H} , but rather defines the magnetic current \mathbf{j}_{mag} . It may be immediately verified that the continuity condition $\nabla \cdot \mathbf{j}_{\text{mag}} = 0$ is fulfilled by all \mathbf{j}_{mag} defined by Eq. (25). As an explicit example, let $B_x = F(y, z) = \sin[k(y + z)]$, et cyclicum. Then, $j_x = (ck/4\pi)\{\cos[k(x + y)] - \cos[k(x + z)]\}$ et cyclicum, where k is in inverse length units.

Summarizing, in a charge-free region described by ME no electric field is internally generated, but there *may* exist a time-independent magnetic background.

B. Non-isolated charge-free region

Consider now a region \mathcal{R}_0 where no charges are present, $\mathcal{Q} = 0$, surrounded by a universe \mathcal{U} where charges do exist. From the superposition principle, total electric field in the region is $\mathbf{E}(\mathcal{R}) = \mathbf{E}(\mathcal{R}_0) + \mathbf{E}(\mathcal{U}) = \mathbf{E}(\mathcal{U})$, where $\mathbf{E}(\mathcal{R}_0)$ denotes the field *internally* generated, and $\mathbf{E}(\mathcal{U})$ represents the field externally produced; from Eq. (7), $\mathbf{E}(\mathcal{R}_0) = 0$. Likewise, for the total magnetic field in the region, $\mathbf{H}(\mathcal{R}) = \mathbf{H}(\mathcal{R}_0) + \mathbf{H}(\mathcal{U})$, where $\mathbf{H}(\mathcal{R}_0)$ is time-independent (see the discussion in previous Subsection A).

It is thus clear that the electric field $\mathbf{E}(\mathcal{R})$ existing inside a charge-free region *is not a free field*; rather, it is generated by charges outside the region. Of course, there is no contradiction with Gauss' law (6) which refers to $\mathbf{E}(\mathcal{U})$ entering and leaving the charge-free region.

C. The simplest charge-neutral universe

Consider a universe containing two equal charges of opposite sign. We can easily obtain from ME with $\varrho = 0$ different solutions $\{\mathbf{E}(\mathcal{U}), \mathbf{H}(\mathcal{U})\}$, depending upon the initial

velocities and separation of the charges.

Consider now a phenomenon that was unknown to Maxwell: charge annihilation. What happens to the electric field $\mathbf{E}(\mathcal{U})$ if the charges meet to annihilate and form two photons? The obvious answer is nothing, the electromagnetic field $\{\mathbf{E}(\mathcal{U}), \mathbf{H}(\mathcal{U})\}$ continues its existence associated to the photons. None the less, there is a difficulty because we are now in situation of $\mathcal{Q} = 0^4$.

So, in a universe populated by two photons there are several fundamental questions to answer. Firstly, do ME apply to them? Let us assume a positive answer. Then, secondly, are we in a charge-free or in a charge-neutral situation? Each possibility has different implications for the inner structure of photons. If photons do not contain charge at all, we are in a charge-free situation where the electric field has disappeared: $\mathbf{E}(\textit{photons}) = 0$ (recall Eq. (7)). Hence, all information about the photons must be contained in the time-dependent magnetic field $\mathbf{H}(\mathcal{U})$. However, as discussed in Subsection A above, in a charge-free region $\mathbf{H}(\mathcal{R}_0)$ is time-independent, which means that the field $\mathbf{H}(\mathcal{U})$ is frozen in time at the moment of annihilation.

Alternatively, if we are in a charge-neutral situation, then the electromagnetic field $\{\mathbf{E}(\mathcal{U}), \mathbf{H}(\mathcal{U})\}$ may continue to exist associated now to the two photons. But then, it means that inside each charge-neutral photon there *must exist* at least a hidden dipole! This interpretation nicely blends with the current view from field theory that attaches electric dipole fields to photons.

IV. CONCLUDING REMARKS

In this paper we argued that a rigorous application of Gauss' law to the solution of Maxwell's equations leads to the identification of two families of solutions: *charge-free* and *charge-neutral*. This immediately implies that electric *free field* does not exist *in the context* of Maxwell's equations.

In an isolated charge-free vacuum, electric field does not exist, but there *may* exist a time-independent background magnetic field. A consideration of the simplest charge-

⁴see point (c) in Section II

neutral universe leads to some interesting conjectures regarding the inner structure of photons.

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